TD 1: Mathematical Prerequisites

These exercises can be found on the wikiversity page
en.wikiversity.org/wiki/Mathematical_prerequisites_for_2d_CFT.

Exercise 1 MICA: Integrating a complex analytic function

Take $a, b \in \mathbb{C}$ and define

$$f(a, b \mid x) = \frac{1}{\left(x^4 + a^4\right)\left(x^2 + b^2\right)^2} \quad , \quad g(a, b) = \int_{-\infty}^{\infty} f(a, b \mid x) dx$$

1. What are the poles and residues of f as a function of x?
2. Compute $g(a, b)$ and discuss its analytic properties.

Exercise 2 MARE: A Lie algebra and its representations

Consider a finite-dimensional Lie algebra $\mathfrak{g}$, with a basis t^a obeying commutation relations $\left[t^a, t^b\right] = f_c^{ab} t^c$. For ρ a representation of $\mathfrak{g}$, we define

$$g^{ab} = \operatorname{Tr}_\rho\left(t^a t^b\right) \quad , \quad K = g_{ab}^{-1} t^a t^b$$

assuming the matrix g^{ab} is invertible.

1. Show that K belongs to the center of the universal enveloping algebra of $\mathfrak{g}$.
2. Compute K for $\mathfrak{g} = \mathfrak{sl}_2$ and $\rho = R_2$ the fundamental representation, i.e. the irreducible representation of dimension 2 . Use a basis J^0, J^+, J^-such that $\left[J^0, J^\pm\right] = \pm J^\pm$and $[J^+, J^-] = 2J^0$.
3. For which values of $j \in \mathbb{C}$ does $\mathfrak{sl}_2$ have an irreducible representation V_j where J^0 has the eigenvalues $\operatorname{Spec}_{V_j}\left(J^0\right) = j - \mathbb{N}$?
4. Compute the value of K in R_2 and V_j. Diagonalize K and J^0 in $R_2 \otimes V_j$, and deduce $R_2 \otimes V_j = \oplus_\pm V_{j \pm \frac{1}{2}}$.
5. By induction on $k \in \mathbb{N}$, decompose $R_2^{\otimes k}$ into irreducible representations. This should include an irreducible representation R_{k+1} of dimension $k+1$. Compute $\operatorname{Spec}_{R_k}\left(J^0\right)$, compute $R_{k_1} \otimes R_{k_2}$ and compute $R_k \otimes V_j$.