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TD 1: Mathematical Prerequisites

These exercises can be found on the wikiversity page
en.wikiversity.org/wiki/Mathematical_prerequisites_for_2d_CFT.

Exercise 1 MICA: Integrating a complex analytic function

Take $a, b \in \mathbb{C}$ and define

$$f(a, b \mid x) = \frac{1}{\left(x^4 + a^4\right)\left(x^2 + b^2\right)^2} \quad , \quad g(a, b) = \int_{-\infty}^{\infty} f(a, b \mid x) dx$$

1. What are the poles and residues of f as a function of x?
2. Compute $g(a, b)$ and discuss its analytic properties.

Answer of exercise 1

1. f has poles at

$$P_a = e^{i\frac{\pi}{4}}\{a, ia, -a, -ia\}$$
$$P_b = \{\pm ib\}$$

- Residues at $p \in P_a$:

$$\mathrm{Res}_p\, f = \frac{1}{\prod_{p' \neq p}(p - p')(p^2 + b^2)^2}$$

and $\prod_{p' \neq p}(p - p') = (p - (-p))(p - ip)(p + ip) = 4p^3$

- Residues at $p \in P_b$: double pole so we use

$$a_{-1} = ((z - p)^2 f)'(p)$$

We get

$$\begin{aligned}\mathrm{Res}_p\, f &= \left(\frac{1}{(x^4 + a^4)(x + p)^2}\right)'(p) \\ &= -\frac{5b^4 + a^4}{4p^3(a^4 + b^4)^2}\end{aligned}$$

2. We integrate f on a half-circle in the upper-half plane. There are cuts along the lines $a \in \mathbb{R}e^{i\frac{\pi}{4}} \cup \mathbb{R}e^{i\frac{3\pi}{4}}$, and $b \in i\mathbb{R}$, where the integral diverges.

If a in the right-quadrant, i.e. $\Re a < \Im a$ and $\Re a > -\Im a$, then we pick residues at $e^{i\frac{\pi}{4}}a, e^{i\frac{\pi}{4}}ia$, and if b is in the right half plane, we pick the residue at ib. Then it is just a matter of summing the residues. If a crosses the cut clockwise, then we go from residues a a, ia to residues at $-ia, a$. The residue in b is continuous. Thus there is a jump of

$$\left(\mathrm{Res}_{iae^{i\frac{\pi}{4}}} - \mathrm{Res}_{-iae^{i\frac{\pi}{4}}}\right) f = \text{something known and computable}$$

Exercise 2 MARE: A Lie algebra and its representations

Consider a finite-dimensional Lie algebra $\mathfrak{g}$, with a basis t^a obeying commutation relations $\left[t^a, t^b\right] = f^{ab}_c t^c$. For ρ a representation of $\mathfrak{g}$, we define

$$g^{ab} = \mathrm{Tr}_\rho \left(t^a t^b\right) \quad , \quad K = g^{-1}_{ab} t^a t^b$$

assuming the matrix g^{ab} is invertible.

1. Show that K belongs to the center of the universal enveloping algebra of $\mathfrak{g}$.
2. Compute K for $\mathfrak{g} = \mathfrak{sl}_2$ and $\rho = R_2$ the fundamental representation, i.e. the irreducible representation of dimension 2 . Use a basis J^0, J^+, J^-such that $\left[J^0, J^\pm\right] = \pm J^\pm$and $[J^+, J^-] = 2J^0$.
3. For which values of $j \in \mathbb{C}$ does $\mathfrak{sl}_2$ have an irreducible representation V_j where J^0 has the eigenvalues $\mathrm{Spec}_{V_j}\left(J^0\right) = j - \mathbb{N}$?
4. Compute the value of K in R_2 and V_j. Diagonalize K and J^0 in $R_2 \otimes V_j$, and deduce $R_2 \otimes V_j = \oplus_\pm V_{j\pm\frac{1}{2}}$.
5. By induction on $k \in \mathbb{N}$, decompose $R_2^{\otimes k}$ into irreducible representations. This should include an irreducible representation R_{k+1} of dimension $k+1$. Compute $\mathrm{Spec}_{R_k}\left(J^0\right)$, compute $R_{k_1} \otimes R_{k_2}$ and compute $R_k \otimes V_j$.

Answer of exercise 2

1. Note that g is the Killing form. Indeed, it is symmetric and $\mathfrak{g}$-invariant:

$$\begin{aligned} g(U, [V, W]) &= \mathrm{Tr}(UVW - UWV) = \mathrm{Tr}(UVW - VUW + VUW - UWV) \\ &= g([U, V], W) + \mathrm{Tr}(VUW - UWV) = g([U, V], W). \end{aligned}$$

   Define the dual basis $t_a = g^{-1}_{ab} t^b$. Take $T \in \mathfrak{g}$, write

$$\begin{aligned} [T, t^a] &= \sum \alpha^a_b t^b \\ [T, t_a] &= \sum \beta^b_a t_b. \end{aligned}$$

   The idea is that $\mathfrak{g}$-invariance implies $\alpha^a_b = -\beta^a_b$.

2. The basis is

$$J_0 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad J^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad J^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

   Which gives $g^{00} = \frac{1}{2}, g^{-+} = 1$. The inverse is $g_{00} = 2, g_{-+} = 1$. Hence $K = 2(J^0)^2 + J^-J^+ + J^+J^- = \frac{3}{2}\,\mathrm{Id}$.

3. J^0 has spectrum $j - \mathbb{N}$ for $j \notin \frac{1}{2}\mathbb{N}^*$. Indeed, if we denote $|j, m\rangle$ the usual basis for the representations of $\mathfrak{sl}_2$, for $j \in \frac{1}{2}\mathbb{N}$, the vector $|j, -j-1\rangle$ is a highest weight. Hence, the complementary of the representation it generates is a subrepresentation, and it is of finite dimension since it is spanned by $|j, m\rangle, m \in \{-j, -j+1, \ldots, j\}$. This basis is an eigenbasis of J^0 and hence the spectrum of J^0 in this representation is $\{-j, -j+1, \ldots, j\}$.

4. In R_2 we have seen $K = \frac{3}{2}$. For V_j, we write

$$K = 2J^0J^0 + J^-J^+ + J^+J^- = 2\left(J^0(J^0+1) + J^-J^+\right)$$

   We only need to compute the action of K on a single vector since we know K acts proportionnally to the identity on any irreducible representation. The easiest thing to do is to compute the value of K on the highest weight $|j,j\rangle$. This vector is killed by J^+ and the value of J^0 is j, so $K|j,j\rangle = 2j(j+1)$.

K and J^0 commute with each other so they can be diagonalised in a common basis. The basis $\{|\pm\frac{1}{2}\rangle \otimes |j,m\rangle\}$ diagonalises J^0, which has eigenvalues $j+\frac{1}{2}$ with multiplicity 1 and $j-\frac{1}{2}-\mathbb{N}$ with multiplicity 2. The eigenvector of $j+\frac{1}{2}$ is a highest weight in the tensor product, and K has eigenvalue $\frac{1}{2}(2j+1)(2j+3)$ on it and the subrepresentation it generates. There is another highest weight, at level $j-\frac{1}{2}$, so there K has eigenvalue $2(j-\frac{1}{2})(j+\frac{1}{2})$. Hence the decomposition.

5. Start by $R_2 \otimes R_2 = R_3 + R_1$, which we obtain by looking at the highest weights as in the previous question (note $R_2 = V_{\frac{1}{2}}/V_{-\frac{3}{2}}$) Then $R_2 \otimes R_k = R_{k-1} + R_{k+1}$, then by induction

$$R_2^{\otimes k} = \bigoplus_{j=0,\frac{1}{2},\dots\frac{k}{2}} m_j R_{2j+1}$$
$$m_j = \binom{k}{\frac{k}{2}-j}$$

The spectrum of J^0 on R_k is $\{\frac{k}{2}, \frac{k-2}{2}, \dots, -\frac{k}{2}\}$.

The idea for $R_{k_1} \otimes R_{k_2}$ is to use associativity of the tensor product. For instance

$$R_2 \otimes R_2 = R_3 \oplus R_1$$

so we can compute the tensor product with R_3 as $R_2 \otimes R_2 - R_1$. We arrive at

$$R_{k_1} \otimes R_{k_2} = \bigoplus_{j \overset{2}{=} |k_1-k_2|+1}^{k_1+k_2-1} R_j$$

And a similar formula for $R_k \otimes V_j$.