TD 2: Conformal Symmetry

These exercises can be found on the wikiversity page
en.wikiversity.org/wiki/Mathematical_prerequisites_for_2d_CFT.

Exercise 1 Questions

1. Show that the scale factor Ω of a conformal transformation, to the power d, coincides with the Jacobian of that transformation.
2. Show that the inversion is a conformal transformation. Write a special conformal transformation in terms of a translation and inversions. Deduce that the special conformal transformation is indeed conformal.
3. Show that two tori whose moduli are related by $\tau \mapsto \frac{a\tau+b}{c\tau+d}$ for

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{Z}) \tag{1}$$

   in $\mathrm{PSL}_2(\mathbb{Z})$ are conformally equivalent.
4. Show that the Polyakov action is invariant under Weyl and conformal transformations.

Exercise 2 COGS: The conformal group of flat space

Consider the Euclidean space $\mathbb{R}^d$ with the flat metric $g_{\mu\nu} = \delta_{\mu\nu}$, and the Minkowski space $\mathbb{R}^{d+1,1}$ with coordinates $Y = (y^\mu, y^-, y^+)$ and the flat metric $\|dY\|^2 = \sum_{\mu=1}^{d} (dy^\mu)^2 - dy^- dy^+$. Consider the diffeomorphisms

$$\varphi : \left\{ \begin{array}{ccc} \mathbb{R}^d & \to & \mathbb{R}^{d+1,1} \\ x^\mu & \mapsto & \left(x^\mu, \|x\|^2, 1\right) \end{array} \right. \quad , \quad \psi : \left\{ \begin{array}{ccc} \mathbb{R}^{d+1,1} & \to & \mathbb{R}^{d+1,1} \\ Y & \mapsto & \frac{1}{y^+} Y = \left(\frac{y^\mu}{y^+}, \frac{y^-}{y^+}, 1\right) \end{array} \right.$$

1. Check that φ is an isometry. Is ψ an isometry? Is it a conformal transformation?
2. Show that the restriction of ψ to the light cone $\mathcal{L} = \left\{ Y \in \mathbb{R}^{d+1,1} \,\middle|\, \|Y\|^2 = 0 \right\}$ is a conformal transformation, and that $\varphi(\mathbb{R}^d) \subset \mathcal{L}$.
3. Let $G \in \mathrm{SO}(d+1,1)$ be an isometry of $\mathbb{R}^{d+1,1}$, in particular G is linear. Show that $\varphi^{-1} \circ \psi \circ G \circ \varphi$ is a conformal transformation of $\mathbb{R}^d$. Deduce that the conformal group of $\mathbb{R}^d$ includes $SO(d+1,1)$.
4. Explicitly write the action of $G \in \mathrm{SO}(d+1,1)$ on x^μ .
5. In the case $d = 2$, find the relation between the two different descriptions of the conformal group: $\mathrm{SO}(3,1)$ and $\mathrm{PSL}_2(\mathbb{C})$.