Paul Roux paul.roux@phys.ens.fr
paul-roux.fr/en/teaching/icfp_cft

TD 2: Conformal Symmetry

These exercises can be found on the wikiversity page
en.wikiversity.org/wiki/Mathematical_prerequisites_for_2d_CFT.

Exercise 1 Questions

1. Show that the scale factor Ω of a conformal transformation, to the power d, coincides with the Jacobian of that transformation.
2. Show that the inversion is a conformal transformation. Write a special conformal transformation in terms of a translation and inversions. Deduce that the special conformal transformation is indeed conformal.
3. Show that two tori whose moduli are related by $\tau \mapsto \frac{a\tau+b}{c\tau+d}$ for

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{Z}) \tag{1}$$

in $\mathrm{PSL}_2(\mathbb{Z})$ are conformally equivalent.

4. Show that the Polyakov action is invariant under Weyl and conformal transformations.

Answer of exercise 1

1. We denote f^*g the pullback of the metric g by the transformation f. The scale factor Ω is defined by $f^*g = \Omega^2 g$. Recall that the expression of the pullback metric is $f^*g = (\mathrm{d}f)^T g \mathrm{d}f$. Taking the determinant we get

$$\det f^*g = \det(\mathrm{d}f)^2 \det g = J^2 \det g \tag{2}$$

$$= \det \Omega^2 g = \Omega^{2d} \det g. \tag{3}$$

where $J = \det \mathrm{d}f$ is the Jacobian of the transformation. Thus

$$J = \Omega^d. \tag{4}$$

2. Denote $f : x \in \mathbb{R}^d \mapsto \frac{1}{||x||^2} x$. The differential is

$$\mathrm{d}f^\mu = \frac{\mathrm{d}x^\mu}{||x||^2} - 2x^\mu x_\nu \frac{\mathrm{d}x^\nu}{||x||^4}$$

Thus if $g = \delta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu$ is the flat metric,

$$\begin{aligned} f^*g &= \delta_{\mu\nu}\mathrm{d}f^\mu \mathrm{d}f^\nu \\ &= \frac{\mathrm{d}x^\mu \mathrm{d}x_\mu}{||x||^4} - 4x^\mu \frac{\mathrm{d}x_\mu}{||x||^2} \frac{x_\nu dx^\nu}{||x||^4} + 4x^\mu x_\mu \left(\frac{x_\nu \mathrm{d}x^\nu}{||x||^4}\right)^2 \\ &= \frac{\mathrm{d}x^\mu \mathrm{d}x_\mu}{||x||^4} \end{aligned}$$

hence we find $\Omega = \frac{1}{||x||^2}$.

3. Let us show that the tori of parameters τ and $\frac{a\tau+b}{c\tau+d}$ are conformally equivalent. By definition, these are the tori

$$T_1 = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$$

$$T_2 = \mathbb{C}/(\mathbb{Z} + \frac{a\tau+b}{c\tau+d}\mathbb{Z}).$$

We will proceed in two steps.

1. Let us first show that

$$(a\tau + b)\mathbb{Z} + (c\tau + d)\mathbb{Z} = \mathbb{Z} + \tau\mathbb{Z}.$$

Because $a, b, c, d \in \mathbb{Z}$, we do have $a\tau+b, c\tau+d \in \mathbb{Z}+\tau\mathbb{Z}$, so $(a\tau+b)\mathbb{Z}+(c\tau+d)\mathbb{Z} \subseteq \mathbb{Z}+\tau\mathbb{Z}$. And because $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{Z})$, the inverse matrix also has integer coefficients so we also have that $1, \tau \in (a\tau + b)\mathbb{Z} + (c\tau + d)\mathbb{Z}$, so that $(a\tau + b)\mathbb{Z} + (c\tau + d)\mathbb{Z} = \mathbb{Z} + \tau\mathbb{Z}$.

2. Next we show that

$$\mathbb{C}/((a\tau + b)\mathbb{Z} + (c\tau + d)\mathbb{Z}) \simeq \mathbb{C}/(\mathbb{Z} + \frac{a\tau + b}{c\tau + d}\mathbb{Z}).$$

Put in this form it is clear that what we need to do is a rescaling. Consider the holomorphic (hence conformal) map

$$\begin{aligned} f : \mathbb{C} &\to \mathbb{C} \\ z &\mapsto (c\tau + d)z \quad , \end{aligned}$$

and denote π the quotient map $\mathbb{C} \to \mathbb{C}/((a\tau + b)\mathbb{Z} + (c\tau + d)\mathbb{Z})$, then $\pi \circ f(z + 1) = \pi((c\tau + d)z + (c\tau + d)) = \pi((c\tau + d)z) = \pi \circ f(z)$, and $\pi \circ f(z + \frac{a\tau+b}{c\tau+d}) = \pi((c\tau + d)z + (a\tau + b)) = \pi((c\tau + d)z) = \pi \circ f(z)$, so $\pi \circ f$ descends to a holomorphic map $\bar{f}$ on the quotient

$$\bar{f} : T_2 = \mathbb{C}/(\mathbb{Z} + \frac{a\tau + b}{c\tau + d}\mathbb{Z}) \to \mathbb{C}/((a\tau + b)\mathbb{Z} + (c\tau + d)\mathbb{Z}).$$

Putting 1) and 2) together we find that

$$T_2 \underset{\bar{f}}{\simeq} \mathbb{C}/((a\tau + b)\mathbb{Z} + (c\tau + d)\mathbb{Z}) = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z}) = T_1.$$

4. The Polyakov action is:

$$\mathcal{S} = \frac{T}{2} \int \mathrm{d}^2\sigma \sqrt{-h} h^{ab} g_{\mu\nu}(X) \partial_a X^\mu(\sigma) \partial_b X^\nu(\sigma)$$

Under a conformal transformation (change of coordinates on the worldsheet), $\sqrt{-h} \mapsto \Omega\sqrt{-h}$, and $\mathrm{d}\sigma \mapsto \Omega^{-1}\mathrm{d}\sigma$, so they compensate. Under a Weyl transformation (change of metric on the worldsheet), $\sqrt{-h} \mapsto \Omega\sqrt{-h}$, and $h^{ab} \mapsto \Omega^{-1}h^{ab}$ (inverse metric). they compensate.

Exercise 2 COGS: The conformal group of flat space

Consider the Euclidean space $\mathbb{R}^d$ with the flat metric $g_{\mu\nu} = \delta_{\mu\nu}$, and the Minkowski space $\mathbb{R}^{d+1,1}$ with coordinates $Y = (y^\mu, y^-, y^+)$ and the flat metric $\|dY\|^2 = \sum_{\mu=1}^{d} (dy^\mu)^2 - dy^- dy^+$. Consider the diffeomorphisms

$$\varphi : \left\{ \begin{array}{ccc} \mathbb{R}^d & \to & \mathbb{R}^{d+1,1} \\ x^\mu & \mapsto & (x^\mu, \|x\|^2, 1) \end{array} \right. \quad , \quad \psi : \left\{ \begin{array}{ccc} \mathbb{R}^{d+1,1} & \to & \mathbb{R}^{d+1,1} \\ Y & \mapsto & \frac{1}{y^+}Y = \left(\frac{y^\mu}{y^+}, \frac{y^-}{y^+}, 1\right) \end{array} \right.$$

1. Check that φ is an isometry. Is ψ an isometry? Is it a conformal transformation?
2. Show that the restriction of ψ to the light cone $\mathcal{L} = \left\{ Y \in \mathbb{R}^{d+1,1} \,\middle|\, \|Y\|^2 = 0 \right\}$ is a conformal transformation, and that $\varphi(\mathbb{R}^d) \subset \mathcal{L}$.
3. Let $G \in \mathrm{SO}(d + 1, 1)$ be an isometry of $\mathbb{R}^{d+1,1}$, in particular G is linear. Show that $\varphi^{-1} \circ \psi \circ G \circ \varphi$ is a conformal transformation of $\mathbb{R}^d$. Deduce that the conformal group of $\mathbb{R}^d$ includes $SO(d + 1, 1)$.

4. Explicitly write the action of $G \in \mathrm{SO}(d+1,1)$ on x^μ .
5. In the case $d = 2$, find the relation between the two different descriptions of the conformal group: $\mathrm{SO}(3,1)$ and $\mathrm{PSL}_2(\mathbb{C})$.

Answer of exercise 2

We denote $h = ||\mathrm{d}Y||^2$ the metric of Minkovski space.

1. Since $\varphi^+ = 1$, $\mathrm{d}\varphi^+ = 0$. Hence

$$\begin{aligned}\varphi^*\eta &= \mathrm{d}\varphi^\mu \mathrm{d}\varphi_\mu - \mathrm{d}\varphi^- \mathrm{d}\varphi^+ \\ &= \mathrm{d}x^\mu \mathrm{d}x_\mu.\end{aligned}$$

The map ψ is not an isometry or a conformal map, since $\psi^* h$ has a term $(\mathrm{d}x^+)^2$.

2. Restricted to the cone of equation $||Y||^2 = 0$, we have $2y_\mu \mathrm{d}y^\mu - y^- \mathrm{d}y^+ - y^+ \mathrm{d}y^- = 0$. Using this and

$$\begin{aligned}\psi^* h &= \mathrm{d}\psi^\mu \mathrm{d}\psi_\mu - \mathrm{d}\psi_- \mathrm{d}\psi^+ = \mathrm{d}\psi^\mu \mathrm{d}\psi_\mu \\ \mathrm{d}\psi^\mu &= \frac{\mathrm{d}y^\mu}{y^+} - \frac{y^\mu \mathrm{d}y^+}{(y^+)^2}\end{aligned}$$

The expression for $\psi^* h$ simplifies and we get that ψ is conformal with $\Omega = \frac{1}{(y^+)^2}$.

3. Let us first show that the transformation is well-defined. Because $\varphi(\mathbb{R}^d) \subseteq \mathcal{L}$, and G preserves the norm, $G \circ \varphi(\mathbb{R}^d) \subseteq \mathcal{L}$. Moreover, $\psi(\mathcal{L}) \subseteq \varphi(\mathbb{R}^d)$, because if $||Y|| = 0$, $y^-/y^+ = y^\mu y_\mu/(y^+)^2$. Hence $\psi \circ G \circ \varphi(\mathbb{R}^d) \subseteq \varphi(\mathbb{R}^d)$, so $\Psi := \varphi^{-1} \circ \psi \circ G \circ \varphi$ is well-defined.

The transformation Ψ is conformal: indeed, φ and G are isometries so in particular they are conformal transformations, and ψ is conformal on the image of $G \circ \varphi$. This means we have constructed a map

$$\begin{aligned}\Psi : \mathrm{SO}(d+1,1) &\to \mathrm{Conf}(\mathbb{R}^d) && (5)\\ G &\mapsto \varphi^{-1} \circ \psi \circ G \circ \varphi && (6)\end{aligned}$$

To check that $\mathrm{Conf}(\mathbb{R}^d)$ indeed contains $\mathrm{SO}(d+1,1)$ as a subgroup, we must check that this map is an injective morphism. The injectivity is immediate, and the fact that it is a morphism follows from the identity $\psi G \psi = \psi G$.

In fact, to show that the conformal group of flat space is exactly $SO(d+1,1)$, there just remains to show that the two groups have the same dimensions, which follows quite directly from the lecture.

4. The action of G on x^μ is

$$\begin{aligned}G \cdot x^\mu &= \varphi^{-1} \circ \psi \circ G(x^\mu, ||x||^2, 1) && (7)\\ &= \varphi^{-1} \circ \psi((G^{a,\nu} x^\nu + G^{a,-}||x||^2 + G^{a,+})_{a \in \{1,\dots,d,\pm\}}) && (8)\\ &= \frac{G^{\mu,\nu} x^\nu + G^{\mu,-}||x||^2 + G^{\mu,+}}{G^{+,\nu} x^\nu + G^{+,-}||x||^2 + G^{++}}. && (9)\end{aligned}$$

5. We want to construct an isomorphism $\mathrm{SO}(3,1) \simeq \mathrm{PSL}_2(\mathbb{C}) = \frac{\mathrm{SL}_2(\mathbb{C})}{(\mathbb{Z}/2\mathbb{Z})}$. The construction is very similar to the construction of the isomorphism $\mathrm{SO}(3) = \frac{\mathrm{SU}(2)}{(\mathbb{Z}/2\mathbb{Z})}$. Indeed $SL(2,\mathbb{C})$ is the complexification of $SU(2)$ (in particular the complexification of the Lie algebra $\mathfrak{su}(2)$ is $\mathfrak{sl}(2)$. The following trick helps: we use the isomorphism

$$\begin{aligned}\mathbb{R}^{1,3} &\xrightarrow{\sim} \mathbb{H} = \{M \in M_2(\mathbb{C}), M^\dagger = M\} \\ (t,x,y,z) &\mapsto \begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix}\end{aligned}$$

and note that $||\cdot||_{\mathbb{R}^{1,3}} = \det(\cdot)$. Then we define

$$\begin{aligned}\varphi : \mathrm{SL}_2(\mathbb{C}) &\to \mathrm{End}(\mathbb{H}) \\ X &\mapsto (M \mapsto XMX^\dagger)\end{aligned}$$

The map φ can be seen as a map to $O(1,3)$ since it preserves the determinant, which is the norm of $\mathbb{R}^{1,3}$:

$$\det XMX^\dagger = \det M \det X \det X^\dagger = \det M \tag{10}$$

since $X \in SL_2(\mathbb{C})$.

It remains to see that φ is surjective, which is not hard. For instance

$$\varphi \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix} \tag{11}$$

is the rotation of angle θ in the x, y plane. Similarly other generators can be obtained. And the kernel of φ is the set of matrices X such that $XMX^\dagger = M$ for all hermitian matrices M, which implies $X \in \{\pm \mathrm{id}\}$, so that φ descends to an isomorphism

$$\mathrm{PSL}_2(\mathbb{C}) \simeq \mathrm{SO}(3,1). \tag{12}$$