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TD 3: Bootstrap approach

These exercises can be found on the wikiversity page
en.wikiversity.org/wiki/Mathematical_prerequisites_for_2d_CFT.

Exercise 1 Questions

1. Show that if the identity field appears in the OPE of two fields, then these two fields are identical.
2. Compute the 2-point and 3-point functions of scalar primary fields.
3. Complete the crossing relation for a 4-point function, by adding the u-channel decomposition, which is deduced from the OPE $\mathcal{O}_1\mathcal{O}_3$. If for all 4-point functions the s-channel and t-channel decompositions coincide, does this imply that the u-channel decomposition agrees as well?
4. Write the generators of the conformal algebra as differential operators, starting with $P_\mu = \frac{\partial}{\partial x^\mu}$ and $D = x^\mu \frac{\partial}{\partial x^\mu}$. Deduce their commutation relations.
5. From reflection positivity, deduce that 2-point functions are positive and 3-point functions are real.

Exercise 2 BOAP: Action of conformal generators on scalar primary fields

Consider a scalar primary field at $x = 0$. Under a conformal transformation that fixes $x = 0$, this field transforms as $\mathcal{O}(0) \to \Omega(0)^{-\Delta}\mathcal{O}(0)$. We would like to deduce the action of the dilations and special conformal generators on this field.

1. Compute the factor $\Omega(x)$ for dilations and special conformal transformations. Deduce $\Omega(0)$.
2. Consider an infinitesimal dilation by a factor $\lambda = 1 + \epsilon$. Show that our primary field transforms as $\mathcal{O}(0) \to \mathcal{O}(0) + \epsilon D \cdot \mathcal{O}(0) + O(\epsilon^2)$ where D is the dilation generator of the conformal algebra, and deduce $D \cdot \mathcal{O}(0)$.
3. Similarly, compute how our field transforms under an infinitesimal special conformal transformation, and deduce $K_\mu \cdot \mathcal{O}(0)$.

Exercise 3 BOSC: Conformal invariance of free scalar fields

1. Consider a massless free scalar field on $\mathbb{R}^d$, of dimension 0. Is the action scale invariant? Is it conformally invariant?
2. If the action is conformally invariant, this means that the classical theory is conformally invariant. Is the quantum theory conformally invariant?
3. If the action is not conformally invariant, modify it so that it becomes conformally invariant, while remaining quadratic. (Hint: use non-integer powers of the Laplacian.) The resulting theory is called mean field theory.

Exercise 4 BOUB: Unitarity bounds on conformal dimensions

We assume that there is an inner product such that the dilation operator is self-conjugate $D^\dagger = D$. We also assume that the inner product is compatible with the Lie algebra structure of the conformal algebra, in the sense that $[A, B]^\dagger = -[B^\dagger, A^\dagger]$.

1. Compute the conjugates of all the generators of the conformal algebra.

2. Starting with a scalar primary state v of conformal dimension Δ, and assuming $(v, v) = 1$, compute the squared norm $(P_\mu v, P_\mu v)$ of the descendant state $P_\mu v$.
3. Assuming unitarity, deduce a bound on Δ.
4. Is this bound valid in the case of the vacuum state?