TD 3: Bootstrap approach

These exercises can be found on the wikiversity page
en.wikiversity.org/wiki/Mathematical_prerequisites_for_2d_CFT.

Exercise 1 Questions

1. Show that if the identity field appears in the OPE of two fields, then these two fields are identical.
2. Compute the 2-point and 3-point functions of scalar primary fields.
3. Complete the crossing relation for a 4-point function, by adding the u-channel decomposition, which is deduced from the OPE $\mathcal{O}_1\mathcal{O}_3$. If for all 4-point functions the s-channel and t-channel decompositions coincide, does this imply that the u-channel decomposition agrees as well?
4. Write the generators of the conformal algebra as differential operators, starting with $P_\mu = \frac{\partial}{\partial x^\mu}$ and $D = x^\mu \frac{\partial}{\partial x^\mu}$. Deduce their commutation relations.
5. From reflection positivity, deduce that 2-point functions are positive and 3-point functions are real.

Answer of exercise 1

1. We use associativity of the OPE: if the identity appears in $V_1(x)V_2(y)$, then V_2 appears in $V_1 \times I$, but since this is just V_1 we get $V_2 = V_1$.
2. The two and three-point functions are fixed by global conformal symmetry. For scalar fields (no spin) we have
   - Two-point: Under a conformal transformation, a two-point function must remain invariant. This implies

$$\langle V_1(x_1)V_2(x_2)\rangle = \left|\frac{\partial x'}{\partial x}\right|_{x=x_1}^{\frac{\Delta_1}{d}} \left|\frac{\partial x'}{\partial x}\right|_{x=x_2}^{\frac{\Delta_2}{d}} \langle V_1(x_1)V_2(x_2)\rangle\,. \tag{1}$$

     Now write $\langle V_{\Delta_1}(x_1)V_{\Delta_2}(x_2)\rangle = f(x_1,x_2)$. We know $f(x_1,x_2) = g(|x_1-x_2|)$ by translation and rotation invariance. If we do a dilation $x \mapsto \lambda x$ we find that g is proportional to $\frac{1}{|x_1-x_2|^{\Delta_1+\Delta_2}}$. There remains to use the invariance under a special conformal transformation $x \mapsto \frac{x-bx^2}{1-2bx+b^2x^2}$. The distance $|x_1-x_2|$ transforms as $|x_1'-x_2'| = \frac{|x_1-x_2|}{(1-2bx_1+b^2x_1^2)^{\frac{1}{2}}(1-2bx_2+b^2x_2^2)^{\frac{1}{2}}}$. And the Jacobian is $\frac{1}{(1-2bx+b^2x^2)^d}$. Hence the conformal invariance implies

$$\frac{1}{|x_1-x_2|^{\Delta_1+\Delta_2}} = \frac{1}{\gamma_1^{\Delta_1}\gamma_2^{\Delta_2}}\frac{(\gamma_1\gamma_2)^{\frac{\Delta_1+\Delta_2}{2}}}{|x_1-x_2|^{\Delta_1+\Delta_2}}, \tag{2}$$

     where $\gamma_i = 1-2bx_i+b^2x_i^2$. Equation (2) is identically satisfied iff $\Delta_1 = \Delta_2$, so

$$\langle V_1(x_1)V_2(x_2)\rangle = \frac{C_{12}}{|z_1-z_2|^{2\Delta_1}}\delta_{\Delta_1,\Delta_2}.$$

   - Three-point: The three-point case works similarly. Rotation and scale invariance implies that

$$\langle V_1V_2V_3\rangle = \sum_{a,b,c}\frac{C_{123}^{abc}}{z_{12}^a z_{23}^b z_{13}^c}, \tag{3}$$

where the sum is over a, b, c such that $a+b+c = \Delta_1+\Delta_2+\Delta_3$. Under a SCT a term of the sum becomes

$$\frac{C_{123}^{abc}}{\gamma_1^{\Delta_1}\gamma_2^{\Delta_2}\gamma_3^{\Delta_3}} \frac{(\gamma_1\gamma_2)^{\frac{a}{2}}(\gamma_2\gamma_3)^{\frac{b}{2}}(\gamma_1\gamma_3)^{\frac{c}{2}}}{z_{12}^a z_{23}^b z_{13}^c}$$

which is identically the same as (3) iff $a+c = 2\Delta_1, a+b = 2\Delta_2, b+c = 2\Delta_3$, which fixes a, b, c as seen in the lecture.

3. In the u-channel, the decomposition is

$$\langle V_1V_2V_3V_4\rangle(x) = \sum_k C_{13k}C_{k24}\mathcal{G}^{(u)}(x).$$

If the decompositions for all correlations in the s and t-channels are verified then the u-channel one follows automatically, by permutation of the fields. However if we only know equality of the (s) and (u) channels for a particular 4 point function with a set order of the fields, then we cannot use permutation invariance.

4. $M_{\mu\nu}$ is the generator of rotations and boosts, so $M_{\mu\nu} = x_\nu\partial_\mu - x_\mu\partial_\nu$. The generator of translations is $P_\mu = \partial_\mu$, and the generator of SCT is (differentiate the transformation with respect to b, at $b=0$: $K_\mu = -x^2\partial_\mu + 2x_\mu x^\nu\partial_\nu$. Computing the brackets is then a simple computation. Here are a few examples:

$$\begin{aligned}
[P_\mu, D] &= \partial_\mu(x^\nu\partial_\nu) - x^\nu\partial_\nu\partial_\mu = \partial_\mu = P_\mu \\
[P_\mu, K_\nu] &= \partial_\mu(-x^2\partial_\nu + 2x_\nu x^\rho\partial_\rho) - (-x^2\partial_\nu\partial_\mu + 2x_\nu x^\rho\partial_\rho\partial_\mu) \\
&= \partial_\mu(-x^2)\partial_\nu + 2\partial_\mu(x_\nu x^\rho)\partial_\rho \\
&= -2x^\rho(\partial_\mu x_\rho)\partial_\nu + 2x_\nu(\partial_\mu x^\rho)\partial_\rho + 2(\partial_\mu x_\nu)(x^\rho\partial_\rho) \\
&= 2\eta_{\mu\nu}D + 2M_{\nu\mu}.
\end{aligned}$$

We used $\partial_\mu x^\nu = \delta_\mu^\nu$, $\partial_\mu x_\nu = \eta_{\mu\nu}$.

5. Reflection positivity for two-point:

$$\langle\mathcal{O}(-x_1, x_2, \ldots, x_d)\mathcal{O}(x_1, x_2, \ldots, x_d)\rangle \geq 0 \tag{4}$$

And we know that

$$\langle\mathcal{O}(-x_1, x_2)\mathcal{O}(x_1, x_2)\rangle = \frac{C}{|2x|^{2\Delta_1}}.$$

Hence we get $C \geq 0$, and so the two-point function itself is positive:

$$\langle\mathcal{O}(x)\mathcal{O}(y)\rangle = \frac{C}{|x-y|^{2\Delta_1}} \geq 0.$$

For the three-point function, we cannot really use the axiom. We can try to loosely argue along the lines of $\mathcal{O}^\dagger(x) = \mathcal{O}(\theta(x))$ but I don't see how to deduce that from the reflection positivity axiom. It seems that the only non-trivial statement we can ask to show is that the three-point functions are real in the basis of operators such that the two-point function is positive, for this unitarity should really be needed. Otherwise we can always given any basis take the combinations $\mathcal{O} + \mathcal{O}^\dagger$, $i(\mathcal{O} - \mathcal{O}^\dagger)$ whenever $\mathcal{O}$ is not unitary.

Exercise 2 BOAP: Action of conformal generators on scalar primary fields

Consider a scalar primary field at $x = 0$. Under a conformal transformation that fixes $x = 0$, this field transforms as $\mathcal{O}(0) \to \Omega(0)^{-\Delta}\mathcal{O}(0)$. We would like to deduce the action of the dilations and special conformal generators on this field.

1. Compute the factor $\Omega(x)$ for dilations and special conformal transformations. Deduce $\Omega(0)$.

2. Consider an infinitesimal dilation by a factor $\lambda = 1 + \epsilon$. Show that our primary field transforms as $\mathcal{O}(0) \to \mathcal{O}(0) + \epsilon D \cdot \mathcal{O}(0) + O(\epsilon^2)$ where D is the dilation generator of the conformal algebra, and deduce $D \cdot \mathcal{O}(0)$.

3. Similarly, compute how our field transforms under an infinitesimal special conformal transformation, and deduce $K_\mu \cdot \mathcal{O}(0)$.

Answer of exercise 2

1. For a dilation $\Omega = \lambda$. A special conformal transformation $\psi(x) = \frac{x - ax^2}{|1-ax|^2}$ can be decomposed as $\psi = IT_{-a}I$ where $I : x \mapsto \frac{x}{|x|^2}$, $T_a : x \mapsto x - a$. The scale factor of I is $\Omega(x) = \frac{1}{x^4}$, and T_a is an isometry, so the scale factor of ψ is $(1 - ax)^4$.
2. Recall that the dilation is the exponential of the infinitesimal dilation.
3. Similar.

Exercise 3 BOSC: Conformal invariance of free scalar fields

1. Consider a massless free scalar field on $\mathbb{R}^d$, of dimension 0. Is the action scale invariant? Is it conformally invariant?
2. If the action is conformally invariant, this means that the classical theory is conformally invariant. Is the quantum theory conformally invariant?
3. If the action is not conformally invariant, modify it so that it becomes conformally invariant, while remaining quadratic. (Hint: use non-integer powers of the Laplacian.) The resulting theory is called mean field theory.

Answer of exercise 3

1. The action is scale & conformally invariant in 2D and only in 2D
2. In general there could be anomalies. However since the theory is free we know all of the correlations, they are given by Wick contractions, and we can check by hand that they are conformally invariant.
3. Go to Fourier space and modify the k^2 term in the Laplacian to an appropriate power.

Exercise 4 BOUB: Unitarity bounds on conformal dimensions

We assume that there is an inner product such that the dilation operator is self-conjugate $D^\dagger = D$. We also assume that the inner product is compatible with the Lie algebra structure of the conformal algebra, in the sense that $[A, B]^\dagger = -[B^\dagger, A^\dagger]$.

1. Compute the conjugates of all the generators of the conformal algebra.
2. Starting with a scalar primary state v of conformal dimension Δ, and assuming $(v, v) = 1$, compute the squared norm $(P_\mu v, P_\mu v)$ of the descendant state $P_\mu v$.
3. Assuming unitarity, deduce a bound on Δ.
4. Is this bound valid in the case of the vacuum state?

Answer of exercise 4

1. Go through the commutators. You find $P_\mu^\dagger = K_\mu$
2. Pass the P^μ to the right, it turns into a K_μ. Then turn this into a commutator because K_μ kills the primary state. Since the field is scalar it is killed by $M_{\mu\nu}$, so

$$(P_\mu v, P_\mu v) = 2\eta_{\mu\mu}\Delta(v, v) \tag{5}$$

3. This is strictly positive if $P_\mu v \neq 0$, so we get $\Delta > 0$
4. No because the vacuum state has $P_\mu v = 0$.