Paul Roux paul.roux@phys.ens.fr
paul-roux.fr/en/teaching/icfp_cft

# TD 4: Fundamental structures of 2D CFT

These exercises can be found on the wikiversity page
en.wikiversity.org/wiki/Mathematical_prerequisites_for_2d_CFT.

## The Virasoro algebra and its representations

### Exercise 1 Questions

1. Does the Virasoro algebra's central term affect global conformal transformations?
2. Write a basis of a Verma module's level 5.
3. From its explicit expression, check that a level-3 null vector is annihilated by $L_1$ and $L_2$. Is it necessary to check it for $L_3$ as well?
4. In the case $c = 28$ find the levels of all the null vectors in the Verma module with dimension $\Delta_{(3,3)}$. How many degenerate quotients does this Verma module have?

## Fields and OPEs

### Exercise 2 Questions

1. Given two CFTs, each one with its own Virasoro algebra and spectrum, let the product CFT be a CFT whose spectrum is the tensor product of the two spectrums. Which Virasoro algebra describes conformal symmetry in the product CFT? What is its central charge?
2. For $V_\Delta$ a primary field, write the OPE of the energy-momentum tensor $T$ with $L_{-2}V_\Delta$, and compare with the OPE of $T$ with itself.
3. Is $C_{12}^k(z_1, z_2)$ related to $C_{21}^k(z_2, z_1)$? To $C_{12}^k(z_2, z_1)$?
4. Assuming the coefficients $f^L$ are known, compute the first few orders of the OPEs $L_{-1}V_{\Delta_1}(z_1)V_{\Delta_2}(z_2)$ and $V_{\Delta_1}(z_1)L_{-1}V_{\Delta_2}(z_2)$.

### Exercise 3 FOGO

In a CFT with local conformal symmetry, we recall that the contribution of $V_\Delta$ and its descendants in an OPE of 2 primary fields reads

$$V_{\Delta_1}(z_1)V_{\Delta_2}(z_2) \supset C^{\Delta}_{\Delta_1,\Delta_2} z_{12}^{\Delta-\Delta_1-\Delta_2}\left(V_\Delta(z_2) + \sum_{L\in\mathcal{L}\setminus\{1\}} z_{12}^{|L|} f^{\Delta,L}_{\Delta_1,\Delta_2} L V_\Delta(z_2)\right) \tag{5}$$

where $\mathcal{L}$ is a basis of Virasoro creation operators.

1. What would be the analogous formula in a CFT with only global conformal symmetry? Show that its universal coefficients are parametrized by integers $k \in \mathbb{N}^*$, and write these coefficients as $\tilde{f}^k = \tilde{f}^{\Delta,k}_{\Delta_1,\Delta_2} = \tilde{f}^{\Delta,L_{-1}^k}_{\Delta_1,\Delta_2}$.
2. Compute the coefficients $\tilde{f}^1$ and $\tilde{f}^2$, and compare them with $f^{L_{-1}}, f^{L_{-1}^2}$.
3. In order to explain why $\tilde{f}^2 \neq f^{L_{-1}^2}$, find how the coefficients $f^L$ behave under a change of basis $L_{-2} \to L_{-2} + \alpha L_{-1}^2$. Which value $\alpha_0$ leads to $\tilde{f}^2 = f^{L_{-1}^2}$?
4. Show that $\left(L_{-2} + \alpha_0 L_{-1}^2\right) V_\Delta$ is a global primary field.

## Exercise 4 2.11 of CFT on the plane

Show that the $T(y)T(z)$ OPE, the commutativity axiom $T(y)T(z) = T(z)T(y)$, and the expansion of $T(y)$ into modes $L_n^{(z_0)}$, imply that such modes obey the Virasoro commutation relations for any choice of $z_0$. To do this, write

$$[L_n^{(z_0)}, L_m^{(z_0)}] = -\frac{1}{4\pi^2}\left(\oint_{z_0} dy \oint_{z_0} dz - \oint_{z_0} dz \oint_{z_0} dy\right)(y-z_0)^{n+1}(z-z_0)^{m+1}T(y)T(z) \ , \tag{12}$$

and use contour manipulations to show that

$$\oint_{z_0} dy \oint_{z_0} dz - \oint_{z_0} dz \oint_{z_0} dy = \oint_{z_0} dy \oint_{y} dz \ . \tag{13}$$

Explain why regular terms in the $T(y)T(z)$ OPE do not contribute to the result.

## Exercise 5 Questions

1. Write the fusion products of the degenerate representation $\mathcal{R}^d_{\langle 3,2\rangle}$ with a Verma module or with another degenerate representation. In which cases are there fewer than 6 terms?
2. At generic central charge, find all subrings of the fusion ring of degenerate representations. Are there any finite-dimensional subrings?
3. What is the smallest Kac table that is not displayed in the article w:minimal models (physics)?

## Exercise 6 2.5 of CFT on the plane

Let us consider the Virasoro algebra with the coupling constant $b^2 = -\frac{q}{p}$ where $p, q$ are strictly positive, coprime integers.

1. Prove the identities

$$\Delta_{\langle r,s\rangle} = \Delta_{\langle r+p,s+q\rangle} = \Delta_{\langle p-r,q-s\rangle} \ . \tag{14}$$

   Under suitable assumptions on $r$ and $s$, show that $\mathcal{V}_{\Delta_{\langle r,s\rangle}}$ has two singular vectors $|\chi_{\langle r,s\rangle}\rangle$ and $|\chi_{\langle p-r,q-s\rangle}\rangle$.

2. Show that each one of the two states $|\chi_{\langle r,s\rangle}\rangle$ and $|\chi_{\langle p-r,q-s\rangle}\rangle$ has a descendant that is itself a singular vector at the level $pq + qr - ps$ in $\mathcal{V}_{\Delta_{\langle r,s\rangle}}$. Assuming that these two singular vectors are in fact identical, enumerate all the singular vectors of $\mathcal{V}_{\Delta_{\langle r,s\rangle}}$.
3. Which ones of these singular vectors are of the type $|\chi_{\langle r',s'\rangle}\rangle$? Show that the singular vector at the level $pq + qr - ps$ is not of this type in $\mathcal{V}_{\Delta_{\langle r,s\rangle}}$, although it is of this type when considered as a singular vector of the Verma modules generated by $|\chi_{\langle r,s\rangle}\rangle$ and $|\chi_{\langle p-r,q-s\rangle}\rangle$.

## Exercise 7 Characters of Virasoro representations

For a representation $\mathcal{R}$ of the Virasoro algebra, let us define the character

$$\mathrm{ch}_{\mathcal{R}}(y) = \mathrm{Tr}_{\mathcal{R}}\, y^{L_0 - \frac{c}{24}} \ . \tag{15}$$

1. Show that the character of a Verma module is

$$\mathrm{ch}_{\mathcal{V}_P}(y) = \frac{y^{-P^2}}{\eta(y)} \ , \tag{16}$$

   where $\eta(y) = y^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-y^n)$ is the Dedekind eta function, and $P$ is the momentum.

2. Deduce that for generic values of the central charge, the character of a maximally degenerate representation is

$$\mathrm{ch}_{\mathcal{R}_{\langle r,s\rangle}} = \frac{y^{-P^2_{\langle r,s\rangle}} - y^{-P^2_{\langle -r,s\rangle}}}{\eta(y)} \ . \tag{17}$$

3. Let us assume $b^2 = -\frac{q}{p}$ where $p, q$ are strictly positive integers, and $1 \leq r \leq p-1$ and $1 \leq s \leq q-1$. Using the results of Exercise 6, show that

$$\mathrm{ch}_{\mathcal{R}_{\langle r,s\rangle}} = \sum_{k\in\mathbb{Z}} \frac{y^{-P^2_{\langle r,s+2qk\rangle}} - y^{-P^2_{\langle r,-s+2qk\rangle}}}{\eta(y)} \ . \tag{18}$$

### Exercise 8

In which minimal models do fusion rules have a $\mathbb{Z}_2$ symmetry like in the Ising case?

## Correlation functions and conformal blocks

### Exercise 9 Questions

1. Assuming we know $Z = \left\langle \prod_{i=1}^N V_{\Delta_i}(z_i) \right\rangle$, compute $Z(y_1, y_2) = \left\langle T(y_1)T(y_2) \prod_{i=1}^N V_{\Delta_i}(z_i) \right\rangle$.
2. Write the generators of the conformal algebra $D, M_{\mu,\nu}, K_\mu, P_\mu$, in terms of Virasoro generators $L_n, \bar{L}_n$.
3. Assuming $\langle V_i V_i V_k \rangle \neq 0$ for some primary fields $V_i, V_k$, what can we say on the conformal spin of $V_k$?

### Exercise 10 Behaviour of the energy-momentum tensor at infinity

If $z$ has dimension $-1$, what is the dimension of $L_{-1}$ according to Eq. [1, eq (2.2.3)]? Then what is the dimension of $T(y)$? Deduce that the differential $T(y)dy^2$ is dimensionless, and should be holomorphic at infinity. Taking $\frac{1}{y}$ to be the natural coordinate at infinity, compare $T(y)dy^2$ with the holomorphic differential $\left(d(\frac{1}{y})\right)^2$, and deduce [1, eq (2.2.8)].

### Exercise 11 Creation operators as differential operators

Check that the representation [1, eq (2.2.15)] of creation operators $L_{-n}^{(z_i)}$ (with $n \geq 1$) as differential operators in $z_1, \cdots, z_N$, is consistent with the commutation relations of the Virasoro algebra.

## References

[1] Sylvain Ribault. Conformal Field Theory on the plane. arXiv: 1406.4290 (1, 7)

[2] Sylvain Ribault. Exactly solvable conformal field theories. 11 2024. (3, 4)