TD 4: Fundamental structures of 2D CFT

These exercises can be found on the wikiversity page
en.wikiversity.org/wiki/Mathematical_prerequisites_for_2d_CFT.

The Virasoro algebra and its representations

Exercise 1 Questions

1. Does the Virasoro algebra's central term affect global conformal transformations?
2. Write a basis of a Verma module's level 5.
3. From its explicit expression, check that a level-3 null vector is annihilated by L_1 and L_2. Is it necessary to check it for L_3 as well?
4. In the case $c = 28$ find the levels of all the null vectors in the Verma module with dimension $\Delta_{(3,3)}$. How many degenerate quotients does this Verma module have?

Answer of exercise 1

1. No when $n, m \in \{-1, 0, 1\}$, the central charge never appears.
2. A basis is given by ordered products of Ls acting on the primary:

$$\{L_{-5}, L_{-4}L_{-1}, L_{-3}L_{-2}, L_{-3}L_{-1}^2, L_{-2}^2L_{-1}, L_{-2}L_{-1}^3, L_{-1}^5\}V_\Delta.$$

3. We use the expressions of the null vectors from [1, eq (1.16)]

$$L_{(3,1)} = -1 + 2\beta^2 L_{-1}^3 - 4\beta^2 L_{-1}L_{-2} + 2\beta^2(\beta^2+1)L_{-3} \tag{1}$$

Multiplying by L_1, L_2, we use $L_i L_{(3,1)}V_\Delta = [L_i, L_{(3,1)}]V_\Delta$ for $i \in \{1, 2\}$, since $L_i V_\Delta = 0$. The commutator can be computed using the Virasoro commutation relations, and in the end we find that $L_i L_{(3,1)} V_\Delta = 0 \Leftrightarrow \Delta = \Delta_{(3,1)}$. Exchanging $\{\beta \leftrightarrow \beta^{-1}\}$ we get the same for $(1, 3)$. For L_3 we do not need to redo any computation since it can be obtained from $[L_1, L_2]$.

4. At rational central charge there can be more than one degenerate field. The central charge $c = 28$ corresponds to $\beta^2 = -2$, hence $\Delta_{(3,3)} = \Delta_{(4,1)}$. There is a null vector at level 9, and one at level 4. The null vector at level 9 has dimension $\Delta_{(3,-3)} = 0 = \Delta_{(1,1)}$,

Fields and OPEs

Exercise 2 Questions

1. Given two CFTs, each one with its own Virasoro algebra and spectrum, let the product CFT be a CFT whose spectrum is the tensor product of the two spectrums. Which Virasoro algebra describes conformal symmetry in the product CFT? What is its central charge?
2. For V_Δ a primary field, write the OPE of the energy-momentum tensor T with $L_{-2}V_\Delta$, and compare with the OPE of T with itself.
3. Is $C_{12}^k(z_1, z_2)$ related to $C_{21}^k(z_2, z_1)$? To $C_{12}^k(z_2, z_1)$?

4. Assuming the coefficients f^L are known, compute the first few orders of the OPEs $L_{-1}V_{\Delta_1}(z_1)V_{\Delta_2}(z_2)$ and $V_{\Delta_1}(z_1)L_{-1}V_{\Delta_2}(z_2)$.

Answer of exercise 2

1. The Virasoro algebra for the product CFT is the tensor product Lie algebra. Central charges are additive; this is seen from the definition of the action of the Lie algebra on the product CFT.

2. The OPE of T with $L_{-2}V_\Delta$ is

$$T(y)L_{-2}V(z) = \frac{[L_2, L_{-2}]V_\Delta}{(y-z)^4} + \frac{[L_1, L_{-2}]V_\Delta}{(y-z)^3} + \frac{L_0 L_{-2}V_\Delta}{(y-z)^2} + \frac{L_{-1}L_{-2}V_\Delta}{y-z} \tag{2}$$

where we have used that $L_2 V_\Delta = L_1 V_\Delta = 0$. We get

$$T(y)L_{-2}V(z) = \frac{(4\Delta + \frac{c}{2})V_\Delta}{(y-z)^4} + \frac{3\partial_z V_\Delta}{(y-z)^3} + \frac{(\Delta+2)L_{-2}V_\Delta}{(y-z)^2} + \frac{\partial_z L_{-2}V_\Delta}{y-z} \tag{3}$$

This matches with the OPE of T with itself if and only if $\Delta = 0$ and $\partial_z V_\Delta = 0$, i.e. if and only if V_Δ is the identity field.

3. The first two are not related, because the field in the RHS of the OPE is evaluated at z_2, so commutativity of the OPE does not give any simple relation. Given that we know the z dependence of C^k_{12}, we find

$$\frac{C^k_{12}(z_1,z_2)}{C^k_{12}(z_2,z_1)} = \left(\frac{z_{12}}{z_{21}}\right)^{\Delta_k - \Delta_1 - \Delta_2} = e^{i\pi(\Delta_k - \Delta_1 - \Delta_2)} \tag{4}$$

4. One simply needs to differentiate the OPE of primary fields with respect to either z_1 or z_2.

Exercise 3 FOGO

In a CFT with local conformal symmetry, we recall that the contribution of V_Δ and its descendants in an OPE of 2 primary fields reads

$$V_{\Delta_1}(z_1)V_{\Delta_2}(z_2) \supset C^\Delta_{\Delta_1,\Delta_2} z_{12}^{\Delta-\Delta_1-\Delta_2}\left(V_\Delta(z_2) + \sum_{L\in\mathcal{L}\setminus\{1\}} z_{12}^{|L|} f^{\Delta,L}_{\Delta_1,\Delta_2} L V_\Delta(z_2)\right) \tag{5}$$

where $\mathcal{L}$ is a basis of Virasoro creation operators.

1. What would be the analogous formula in a CFT with only global conformal symmetry? Show that its universal coefficients are parametrized by integers $k \in \mathbb{N}^*$, and write these coefficients as $\tilde{f}^k = \tilde{f}^{\Delta,k}_{\Delta_1,\Delta_2} = \tilde{f}^{\Delta,L^k_{-1}}_{\Delta_1,\Delta_2}$.

2. Compute the coefficients $\tilde{f}^1$ and $\tilde{f}^2$, and compare them with $f^{L_{-1}}, f^{L^2_{-1}}$.

3. In order to explain why $\tilde{f}^2 \neq f^{L^2_{-1}}$, find how the coefficients f^L behave under a change of basis $L_{-2} \to L_{-2} + \alpha L^2_{-1}$. Which value α_0 leads to $\tilde{f}^2 = f^{L^2_{-1}}$?

4. Show that $\left(L_{-2} + \alpha_0 L^2_{-1}\right) V_\Delta$ is a global primary field.

Answer of exercise 3

1. With only global symmetry we only have L_{-1}. The z-dependence is unchanged because it is derived using global Ward identities. The OPE writes

$$V_{\Delta_1}(z_1)V_{\Delta_2}(z_2) \supset C^\Delta_{\Delta_1,\Delta_2} z_{12}^{\Delta-\Delta_1-\Delta_2}\left(V_\Delta(z_2) + \sum_{k\in\mathbb{N}^*} z_{12}^{k} \tilde{f}^{\Delta,k}_{\Delta_1,\Delta_2} L^k_{-1} V_\Delta(z_2)\right) \tag{6}$$

2. The coefficients are fixed by global conformal symmetry. We do not need to redo any computations, we can simply use the results from the course, except we can only keep the results obtained using the Ward identity associated to L_{-1}. We get

$$L_1\tilde{W}_k = \theta_k \tilde{W}_{k-1} \tag{7}$$

where $\tilde{W}_k = C^k L_{-1}^k V_\Delta$, $\theta_k = k - 1 + \Delta_1 - \Delta_2 + \Delta$. Pulling the L_{-1} through we find

$$\tilde{f}^1 = \frac{\Delta_1 - \Delta_2 + \Delta}{2\Delta} \tag{8}$$

$$\tilde{f}^2 = \frac{\Delta_1 - \Delta_2 + \Delta}{2\Delta}\frac{1 + \Delta_1 - \Delta_2 + \Delta}{2(2\Delta+1)} \tag{9}$$

We see that $\tilde{f}^2 \neq f^{L_{-1}^2}$.

3. The coefficients f^L are the solution of

$$L_{1,2}W_N = \theta_{1,2|N}W_{N-1} \tag{10}$$

under the change of basis W_2 is unchanged if and only if

$$\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}\begin{pmatrix} C'^{L_{-1}^2} \\ C'^{L_{-2}+\alpha L_{-1}^2} \end{pmatrix} = \begin{pmatrix} C^{L_{-1}^2} \\ C^{L_{-2}} \end{pmatrix} \tag{11}$$

Thus $f^{L_{-1}^2} \to f^{L_{-1}^2} - \alpha f^{L_{-2}}$. From the expression [2, eq 1.53b] we find that $f^{L_{-1}^2} - \alpha f^{L_{-2}} = \tilde{f}^2$ if and only if $\alpha = \frac{-3}{2(2\Delta+1)}$. This can be found we little computation by first cancelling the term in $\Delta + 2\Delta_1 - \Delta_2$. It is then not hard to check that this gives the right value for $f^{L_{-1}^2} - \alpha f^{L_{-2}}$.

4. It is an immediate computation, just check that applying L_1 gives zero. This choice of basis is a posteriori somewhat natural, since the global primary sitting at level 2 is the only field other that L_{-1}^2 that is "special" with respect to global conformal symmetry.

Exercise 4 2.11 of CFT on the plane

Show that the $T(y)T(z)$ OPE, the commutativity axiom $T(y)T(z) = T(z)T(y)$, and the expansion of $T(y)$ into modes $L_n^{(z_0)}$, imply that such modes obey the Virasoro commutation relations for any choice of z_0. To do this, write

$$[L_n^{(z_0)}, L_m^{(z_0)}] = -\frac{1}{4\pi^2}\left(\oint_{z_0} dy \oint_{z_0} dz - \oint_{z_0} dz \oint_{z_0} dy\right)(y-z_0)^{n+1}(z-z_0)^{m+1}T(y)T(z)\ , \tag{12}$$

and use contour manipulations to show that

$$\oint_{z_0} dy \oint_{z_0} dz - \oint_{z_0} dz \oint_{z_0} dy = \oint_{z_0} dy \oint_{y} dz\ . \tag{13}$$

Explain why regular terms in the $T(y)T(z)$ OPE do not contribute to the result.

Answer of exercise 4

The subtlety is that the products are always implicitely time ordered. We can fix z and integrate over y. Then we get the integral over a circle with radius $> |z|$, minus an integral over a circle with radius $< |z|$. Since the function we are integrating is holomorphic outside of z, this is equal to the integral of the function on a circle around z. Then we can integrate over z. After that it is simply a matter of writing the TT OPE and carefully computing the residues.

Exercise 5 Questions

1. Write the fusion products of the degenerate representation $\mathcal{R}^d_{\langle 3,2\rangle}$ with a Verma module or with another degenerate representation. In which cases are there fewer than 6 terms?
2. At generic central charge, find all subrings of the fusion ring of degenerate representations. Are there any finite-dimensional subrings?
3. What is the smallest Kac table that is not displayed in the article w:minimal models (physics)?

Answer of exercise 5

1. Directly take the formula from [2, (1.65), (1.66)]. There are less than six terms when we fuse with a degenerate field with $r \leq 2$ or $s = 1$.
2. The $(V_{<1,s>})_{s\in\mathbb{N}^*}$ are stable, $V_{<1,2s+1>}, s \in \mathbb{N}^*$ too. If a subring contains a $V_{<1,s>}$ with s even, by fusing it with itself we get operators with odd s, and then by fusing odd with even we eventually get $V_{<1,2>}$ and so we get all values of s. There are no finite subrings at generic central charge. Similarly if we reverse r, s. By the same argument, $(V_{<2r+1,2s+1>})_{r,s\in\mathbb{N}^*}$ is a subring, and if there is an even r then we get all values of r. In summary the subrings are $V_{<2\mathbb{N}+1,2\mathbb{N}+1>}, V_{<1,2\mathbb{N}+1>}, V_{<2\mathbb{N}+1,1>}, V_{<1,\mathbb{N}>}, V_{<\mathbb{N},1>}$.
3. It is $K_{7,2}$, which has 1 row and 6 columns, with dimensions given by the usual Kac formula.

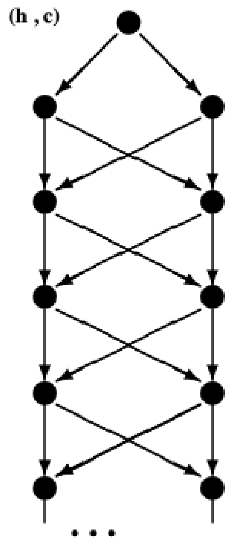
Exercise 6 2.5 of CFT on the plane

Let us consider the Virasoro algebra with the coupling constant $b^2 = -\frac{q}{p}$ where p, q are strictly positive, coprime integers.

1. Prove the identities
$$\Delta_{\langle r,s\rangle} = \Delta_{\langle r+p,s+q\rangle} = \Delta_{\langle p-r,q-s\rangle} \ . \tag{14}$$
Under suitable assumptions on r and s, show that $\mathcal{V}_{\Delta_{\langle r,s\rangle}}$ has two singular vectors $|\chi_{\langle r,s\rangle}\rangle$ and $|\chi_{\langle p-r,q-s\rangle}\rangle$.
2. Show that each one of the two states $|\chi_{\langle r,s\rangle}\rangle$ and $|\chi_{\langle p-r,q-s\rangle}\rangle$ has a descendant that is itself a singular vector at the level $pq + qr - ps$ in $\mathcal{V}_{\Delta_{\langle r,s\rangle}}$. Assuming that these two singular vectors are in fact identical, enumerate all the singular vectors of $\mathcal{V}_{\Delta_{\langle r,s\rangle}}$.
3. Which ones of these singular vectors are of the type $|\chi_{\langle r',s'\rangle}\rangle$? Show that the singular vector at the level $pq + qr - ps$ is not of this type in $\mathcal{V}_{\Delta_{\langle r,s\rangle}}$, although it is of this type when considered as a singular vector of the Verma modules generated by $|\chi_{\langle r,s\rangle}\rangle$ and $|\chi_{\langle p-r,q-s\rangle}\rangle$.

Answer of exercise 6

1.
$$\begin{aligned} P_{<p+r,q+s>} &= P_{<r,s>} + pb + qb^{-1} \\ &= P_{<r,s>} + i\sqrt{pq} - i\sqrt{qp} = P_{<r,s>} \\ P_{<p-r,q-s>} &= -P_{<r,s>} \end{aligned}$$
This proves the equality of the Δs. If $rs \neq (p-r)(q-s)$, there are two singular vectors, at levels rs and $(p-r)(q-s)$ (they are at different levels so we know they're distinct).
2. $|\chi_{(r,s)}\rangle$ has dimension $\Delta_{(r,-s)} = \Delta_{(p-r,q+s)}$, so it has a singular vector at level $(p-r)(q+s)$, which is total level $(p-r)(q+s) + rs = pq + sp - rq$. And $|\chi_{(p-r,q-s)}\rangle$ has dimension $\Delta_{(p-r,s-q)} = \Delta_{(r,2q-s)}$ so it has a singular vector at level $r(2q-s)+(p-r)(q-s) = rq+pq-ps$. There can only exist a single singular vector at a given level, hence these two have to be the same. By recursion, we find that the full hierarchy of null vectors looks like this:



Exercise 7 Characters of Virasoro representations

For a representation $\mathcal{R}$ of the Virasoro algebra, let us define the character

$$\mathrm{ch}_{\mathcal{R}}(y) = \mathrm{Tr}_{\mathcal{R}}\, y^{L_0 - \frac{c}{24}} \ . \tag{15}$$

1. Show that the character of a Verma module is

$$\mathrm{ch}_{\mathcal{V}_P}(y) = \frac{y^{-P^2}}{\eta(y)} \ , \tag{16}$$

where $\eta(y) = y^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-y^n)$ is the Dedekind eta function, and P is the momentum.

2. Deduce that for generic values of the central charge, the character of a maximally degenerate representation is

$$\mathrm{ch}_{\mathcal{R}_{\langle r,s\rangle}} = \frac{y^{-P^2_{\langle r,s\rangle}} - y^{-P^2_{\langle -r,s\rangle}}}{\eta(y)} \ . \tag{17}$$

3. Let us assume $b^2 = -\frac{q}{p}$ where p, q are strictly positive integers, and $1 \leq r \leq p-1$ and $1 \leq s \leq q-1$. Using the results of Exercise 6, show that

$$\mathrm{ch}_{\mathcal{R}_{\langle r,s\rangle}} = \sum_{k\in\mathbb{Z}} \frac{y^{-P^2_{\langle r,s+2qk\rangle}} - y^{-P^2_{\langle r,-s+2qk\rangle}}}{\eta(y)} \ . \tag{18}$$

Answer of exercise 7

1. Just note that η is related to the generating function for integer partitions, i.e. $\eta = y^{\frac{1}{24}} \sum_n p(n) y^n$. Since the level n has dimension $p(n)$ and $L_0 = \Delta + n$ there, we get the result.

2. Quotient module = difference of characters.

3. We need to quotient by the modules generated by $|\chi_{(r,s)}\rangle$ and $|\chi_{(p-r,q-s)}\rangle$. We would get

$$\chi = \frac{1}{\eta}(y^{-P^2_{(r,s)}} - y^{-P^2_{(r,-s)}} - y^{-P^2_{(r,-s+2q)}}) \tag{19}$$

but then we quotiented their intersection twice so we need to add it. Their intersection is generated by their common descendent, which has dimension $\Delta_{(r-p,q+s)} = \Delta_{(r,s+2q)}$. Its character can be computed similarly, and so we can do this recursively to get the required formula. I think it should be a sum over $\mathbb{N}$?

Exercise 8

In which minimal models do fusion rules have a $\mathbb{Z}_2$ symmetry like in the Ising case?

Answer of exercise 8

Note that because

$$V^d_{\langle 1,3\rangle} \times V^d_{\langle 1,3\rangle} \supset -V^d_{\langle 1,3\rangle} \tag{20}$$

$$V^d_{\langle 1,3\rangle} \to V^d_{\langle 1,3\rangle} \tag{21}$$

is not an automorphism of the algebra of degenerate fields. Let us see when

$$V^d_{\langle 1,2\rangle} \to -V^d_{\langle 1,2\rangle} \tag{22}$$

$$V^d_{\langle 2,1\rangle} \to V^d_{\langle 2,1\rangle} \tag{23}$$

is an algebra autorphism. First, by fusing $V^d_{\langle 1,2\rangle}$ repeatedly with itself we find that

$$V^d_{\langle 1,s\rangle} \to (-1)^{s+1} V^d_{\langle 1,s\rangle}. \tag{24}$$

Then by fusing with $V^d_{\langle 2,1\rangle}$ we find that

$$V^d_{\langle r,s\rangle} \to (-1)^{s+1} V^d_{\langle r,s\rangle} \tag{25}$$

But in a minimal model

$$V^d_{(r,s)} = V^d_{(p-r,q-s)}. \tag{26}$$

Thus, when q is odd, applying a $\mathbb{Z}_2$ transformation we get a contradiction. Conversely, when q is even, the above $\mathbb{Z}_2$ symmetry is a symmetry of $M_{p,q}$.

Correlation functions and conformal blocks

Exercise 9 Questions

1. Assuming we know $Z = \left\langle \prod_{i=1}^N V_{\Delta_i}(z_i) \right\rangle$, compute $Z(y_1, y_2) = \left\langle T(y_1)T(y_2) \prod_{i=1}^N V_{\Delta_i}(z_i) \right\rangle$.
2. Write the generators of the conformal algebra $D, M_{\mu,\nu}, K_\mu, P_\mu$, in terms of Virasoro generators $L_n, \bar{L}_n$.
3. Assuming $\langle V_i V_i V_k \rangle \neq 0$ for some primary fields V_i, V_k, what can we say on the conformal spin of V_k?

Answer of exercise 9

1. Use the OPE

$$T(y)T(z) = \frac{\frac{c}{2}}{(y-z)^4} + \frac{2T}{(y-z)^2} + \frac{\partial T}{y-z} + O(y-z) \tag{27}$$

Thus

$$Z(y_1, y_2) = \frac{\frac{c}{2}}{(y_1-y_2)^4} Z + \frac{2}{(y_1-y_2)^2} Z(y_2) + \frac{1}{y_1-y_2} \partial Z(y_2) \tag{28}$$

where $Z(y) =< T(y) \prod V_{\Delta_i}(z_i) >$ is given in the lecture notes.

2. $D = L_0 + \bar{L}_0$, $P_t = L_{-1} + \bar{L}_{-1}$, $P_x = \pm i(L_{-1} - \bar{L}_{-1})$, $M_{\mu\nu}$ has only a single component (2D Lorentz group just has rotations) which is $L_0 - \bar{L}_0$. There are also some $\frac{1}{2}$ here and there to check, also maybe the is are slightly wrong.

3. The field V_k must have integer spin since the sum of the spins must be an integer, and $2S_1 \in \mathbb{Z}$. In fact we can get a better result. We know how C_{iik} behaves under a transposition of two fields:

$$C_{ijk} = (-1)^{S_1+S_2+S_3} C_{jik} \tag{29}$$

In the particular case where the two first fields are identical, we find

$$C_{iik} = (-1)^{S_k} C_{iik} \tag{30}$$

hence $S_k \in 2\mathbb{Z}$.

Exercise 10 Behaviour of the energy-momentum tensor at infinity

If z has dimension -1, what is the dimension of L_{-1} according to Eq. [1, eq (2.2.3)]? Then what is the dimension of $T(y)$? Deduce that the differential $T(y)dy^2$ is dimensionless, and should be holomorphic at infinity. Taking $\frac{1}{y}$ to be the natural coordinate at infinity, compare $T(y)dy^2$ with the holomorphic differential $\left(d(\frac{1}{y})\right)^2$, and deduce [1, eq (2.2.8)].

Answer of exercise 10

L_{-1} has dimension 1, and T has dimension $\frac{L_{-1}}{z}$ so dimension 2. Then the form $T(y)\mathrm{d}y^2$ is dimensionless and so, assuming this is a good object on the sphere, nothing specific should happen around ∞, otherwise this point would be special. If we write this 2-form in the coordinate system around infinity it reads

$$T(\frac{1}{y})\mathrm{d}(\frac{1}{y})^2 = T(\frac{1}{y})\frac{\mathrm{d}y^2}{y^4} \tag{31}$$

Hence the function $T(\frac{1}{y})y^{-4}$ is holomorphic around 0 and so the lowest term in $T(\frac{1}{y})$ is in y^4. Said otherwise $T(y) =_{y\to\infty} O(\frac{1}{y^4})$.

Exercise 11 Creation operators as differential operators

Check that the representation [1, eq (2.2.15)] of creation operators $L_{-n}^{(z_i)}$ (with $n \geq 1$) as differential operators in $z_1, \cdots, z_N$, is consistent with the commutation relations of the Virasoro algebra.

Answer of exercise 11

Write $L_{-n} = \frac{-1}{z^{n-1}}\partial + \frac{n-1}{z^n}\Delta$ Then it is a simple matter of writing commutators of derivatives. A more elegant approach is to note that

$$L_{-n} = z^{\Delta}\ell_n z^{-\Delta} \tag{32}$$

where

$$\ell_n = -z^{n+1}\partial_z \tag{33}$$

Then the commutator is automatically the Witt commutator:

$$[z^{\Delta}\ell_{-n}z^{-\Delta}, z^{\Delta}\ell_{-m}z^{-\Delta}] = z^{\Delta}[\ell_{-n}, \ell_{-m}]z^{-\Delta}. \tag{34}$$

References

[1] Sylvain Ribault. Conformal Field Theory on the plane. arXiv: 1406.4290 (1, 7)

[2] Sylvain Ribault. Exactly solvable conformal field theories. 11 2024. (3, 4)