TD 5: Exactly solvable 2D CFTs

These exercises can be found on the wikiversity page
en.wikiversity.org/wiki/Mathematical_prerequisites_for_2d_CFT.

Exercise 1 Questions

1. In the Ising minimal model, what are the OPE spectrums? Write the channel spectrums of all inequivalent 4-point functions.
2. In a generalized minimal model, write the 3 channel spectrums of $< V^d_{\langle 3,3\rangle} V^d_{\langle 2,4\rangle} V^d_{\langle 1,4\rangle} V^d_{\langle 2,5\rangle} >$
3. What are the largest possible non-diagonal spectrums for a CFT with 2 degenerate fields, such that all spins are integer?

Answer of exercise 1

1. The OPE spectrums are $\{V^d_{\langle 2,1\rangle}\}$, $\{V^d_{\langle 1,1\rangle}, V^d_{\langle 3,1\rangle}\}$. The non-zero four-point functions are

$$< (V^d_{\langle 1,2\rangle})^4 > \tag{1}$$
$$< (V^d_{\langle 1,3\rangle})^4 > \tag{2}$$
$$< (V^d_{\langle 1,2\rangle})^2 (V^d_{\langle 1,3\rangle})^2 > \tag{3}$$

With appropriately either $\{V^d_{\langle 2,1\rangle}\}$ or $\{V^d_{\langle 1,1\rangle} V^d_{\langle 3,1\rangle}\}$ in the channel spectra.

2. Let us do it for one channel, say the s-channel. The channel spectrum is the intersection of the fields that appear in the OPEs of $V^d_{\langle 1,4\rangle} \times V^d_{\langle 2,5\rangle}$ and $V^d_{\langle 3,3\rangle} \times V^d_{\langle 2,4\rangle}$. This gives

$$\mathcal{S}^{(s)} = V^d_{\langle 2,2\rangle} + V^d_{\langle 2,4\rangle} + V^d_{\langle 2,6\rangle}. \tag{4}$$

The same argument can be applied to other channels.

3. With two degenerate fields, we must have $r, s \in \frac{1}{2}\mathbb{Z}$. Since additionally $rs \in \mathbb{Z}$, this leaves three possibilities:

$$r \in \frac{1}{2}\mathbb{N}^*, s \in 2\mathbb{Z} \tag{5}$$
$$r \in 2\mathbb{Z}, s \in \frac{1}{2}\mathbb{N}^* \tag{6}$$
$$r \in \mathbb{N}^*, s \in \mathbb{Z} \tag{7}$$

Exercise 2 ESDI: conformal dimensions of non-diagonal fields

We consider the non-diagonal primary fields with integer conformal spins, in a CFT that has a degenerate field $V^d_{\langle 1,3\rangle}$. We assume $\Re\beta^2 > 0$.

1. Which non-diagonal fields $V_{(r,s)}$ can exist?
2. Show that the set of their total conformal dimensions $\{\Delta + \bar{\Delta}\}$ is discrete and bounded from below.
3. Show that there are finitely many fields whose total dimension is below any given bound.
4. Assuming $\beta^2 \in \mathbb{R}$, sketch the set of non-diagonal fields on a plot whose axes are $\Delta - \bar{\Delta}$ and $\Delta + \bar{\Delta}$.

Answer of exercise 2

1. As we said above, all the $V_{(r,s)}$ with $r \in \frac{1}{2}\mathbb{N}^*, s \in \frac{1}{r}\mathbb{Z}$.
2.
$$\Re(\Delta + \bar{\Delta}) = \Re(\frac{c-1}{12} + \frac{1}{2}(r^2\beta^2 + s^2\beta^{-2})) \tag{8}$$
$$\geq \Re(\frac{c-1}{12}) \tag{9}$$
so the spectrum is bounded from below.
3. See q.4
4. At fixed spin $= S$ the set of $\Delta + \bar{\Delta}$ is $\frac{c-1}{12} + \frac{1}{2}(r^2\beta^2 + \frac{S^2}{r^2}\beta^{-2})$. This is an increasing function of r, convex, so there are only a finite number of these numbers under a given bound. Moreover, if S is large enough, $\Delta + \bar{\Delta}$ is above any bound. So there are only a finite number of possible S for $\Delta + \bar{\Delta}$ to be under a given bound. Thus there are a finite number of fields under any bound. On a plot, we have a parabolic envelope ($\Delta + \bar{\Delta}$ grows quadratically with S) under which there are no points, and then on each vertical integer line we have discrete points, starting on the parabola, and more and more spaced as r increases.

Exercise 3 ESDS: D-series minimal models

1. Find the spectrums of D-series minimal models in Wikipedia.
2. What is the intersection of the spectrums of the A-series and D-series minimal models at the same central charge? Is this the spectrum of a consistent CFT?
3. Show that the spectrum of the D-series minimal model $\text{DMM}_{6,5}$ is a subset of the spectrum of the 3-state Potts CFT. Explain why the diagonal primary field $V_{P_{(0,\frac{1}{2})}}$ appears in the non-diagonal sector of $\text{DMM}_{6,5}$. Admitting that $V_{P_{(0,\frac{1}{2})}}$ transforms in the standard representation of the symmetric group S_3, explain why it appears twice in $\text{DMM}_{6,5}$.