TD 6: Analytic bootstrap

These exercises can be found on the wikiversity page
en.wikiversity.org/wiki/Mathematical_prerequisites_for_2d_CFT.

1 Belavin-Polyakov-Zamolodchikov differential equations

See Section 3.1 of Ref. [2]

Exercise 1 Questions

1. For V_i primary fields, we would like to compute $\langle L_{-3}V_1(z)V_2(0)V_3(\infty)V_4(1)\rangle$ as a differential operator acting on $\langle V_1(z)V_2(0)V_3(\infty)V_4(1)\rangle$. To do this, we may use integrals of the type $\oint_{z_0} dy \frac{y(y-1)(y-a)}{(y-z)^2} \langle T(y)V_1(z)V_2(0)V_3(\infty)V_4(1)\rangle$ with $z_0 \in \{0,1,\infty,z\}$. Which value of a allows us to avoid having a contribution of L_{-2}? Deduce the desired result.
2. How do hypergeometric blocks behave under field permutations $V_i(z_i) \leftrightarrow V_j(z_j)$? Under reflections of momentums $P_i \to -P_i$?
3. Compute the inverse of the degenerate fusing matrix F, and compare it to F.
4. Calling $F^{s\to t}$ the degenerate fusing matrix that relates s-channel to t-channel blocks, compute $F^{s\to u}$ and $F^{t\to u}$, and check $F^{s\to t}F^{t\to u} = F^{s\to u}$.

Exercise 2 Third-order BPZ equation

Write the BPZ equation for an N-point function involving a degenerate field $V_{\langle 1,3\rangle}(x)$. In the case $N = 3$, rederive the relevant fusion rule.

Exercise 3 BPZ equations from fusion rules

Let us assume that we do not know the structures of the Virasoro algebra or its representations, but only the fusion rules. We are interested in the four-point function $\mathcal{F}(x) = \left\langle V_{\langle 2,1\rangle}(x)V_{\Delta_1}(0)V_{\Delta_2}(\infty)V_{\Delta_3}(1)\right\rangle$, where the degenerate field $V_{\langle 2,1\rangle}$ is defined by the fusion rule

$$\boxed{\mathcal{R}_{\langle 2,1\rangle} \times \mathcal{V}_P = \sum_\pm \mathcal{V}_{P\pm\frac{b}{2}}} \quad , \quad \boxed{\mathcal{R}_{\langle 1,2\rangle} \times \mathcal{V}_P = \sum_\pm \mathcal{V}_{P\pm\frac{1}{2b}}} \; . \tag{11}$$

Since this fusion rule has two terms, we conjecture that $\mathcal{F}(x)$ obeys a second-order differential equation of the type

$$\left\{\frac{\partial^2}{\partial x^2} + a(x)\frac{\partial}{\partial x} + b(x)\right\}\mathcal{F}(x) = 0 \; . \tag{12}$$

1. From the OPEs and fusion rules of $V_{\langle 2,1\rangle}$, deduce that $\mathcal{F}(x)$ has regular singular points at $x = 0, 1, \infty$, and compute its characteristic exponents. Use

$$V_{\Delta_1,\bar\Delta_1}(z_1)V_{\Delta_2,\bar\Delta_2}(\infty) = \sum_{\Delta_3,\bar\Delta_3} (-1)^{\sum S_i} \frac{C_{123}}{B_3} \left|z_1^{\Delta_2-\Delta_1-\Delta_3}\right|^2 \left(V_{\Delta_3,\bar\Delta_3}(\infty) + O\left(z_1^{-1}\right)\right) . \tag{13}$$

for the OPE $V_{\langle 2,1\rangle}(x)V_{\Delta_2}(\infty)$.

2. Using the ansatz $\mathcal{F}(x) \underset{x\to 0}{=} x^\lambda(1+O(x))$, show that $a(x)$ has a simple pole at $x=0$, and that $b(x)$ has a double pole. Using the ansatz $\mathcal{F}(x) \underset{x\to\infty}{=} x^\lambda(1+O(x^{-1}))$, show that near $x=\infty$ we have $a(x)=O(x^{-1})$ and $b(x)=O(x^{-2})$. Assuming that $a(x)$ and $b(x)$ are meromorphic functions with no poles outside $\{0,1,\infty\}$, write these functions in terms of five coefficients, and deduce that the six characteristic exponents obey the relation

$$\sum_\pm \left(\lambda_\pm^{(0)} + \lambda_\pm^{(1)} - \lambda_\pm^{(\infty)}\right) = 1 \ . \tag{14}$$

3. Show that this relation is satisfied by the characteristic exponents that follow from the fusion rules, and compute the functions $a(x)$ and $b(x)$. Compare the resulting differential equation with the BPZ equation

$$\left\{\frac{x(1-x)}{b^2}\frac{\partial^2}{\partial x^2} + (2x-1)\frac{\partial}{\partial x} + \Delta_{\langle 2,1\rangle} + \frac{\Delta_1}{x} - \Delta_2 + \frac{\Delta_3}{1-x}\right\}\mathcal{F}(x) = 0 \ . \tag{15}$$

4. Show that the third-order BPZ equation is not completely determined by its characteristic exponents. In particular, show that adding a term proportional to $\frac{1}{x^2(x-1)^2}$ to the differential operator $\frac{\partial^3}{\partial x^3} + \cdots$ does not affect the characteristic exponents.

Exercise 4 ABOF: Degenerate 4-point functions with one propagating field

We consider a 4-point function of the type $\left\langle V^d_{\langle 2,1\rangle} V_P V^d_{\langle r,s\rangle} V_{P'}\right\rangle$.

1. For which values of P' is the s-channel spectrum made of only one primary field? Choose one such value.
2. Compute the t-channel and u-channel spectrums.
3. Compute the corresponding degenerate fusing matrix.
4. Write the relations between all 4-point structure constants.

2 Shift equations for structure constants

Exercise 5 Questions

1. Check that any 4-point structure constant is invariant under renormalizations of the channel field $V_k \to \lambda_k V_k$. Furthermore, check that any ratio of 4-point structure constants is invariant under any field renormalization $V_i \to \lambda_i V_i$ with $i \in \{1,2,3,4\}$.
2. How do shift equations behave under parity? i.e. under the exchange of left and right momentums $\forall i, P_i \leftrightarrow \bar{P}_i$.
3. In how many ways can the ratio $\frac{C_{1-23+}}{C_{1+23-}}$ be computed using 2 shift equations? Check that the different computations yield the same result.

Exercise 6 ABUD: Counting independent structure constants

Given 3 numbers $r_i \in \frac{1}{2}\mathbb{N}^*$ such that $r_1+r_2+r_3 \in \mathbb{N}$, we are interested in structure constants of the type $C_{(r_1,s_1)(r_2,s_2)(r_3,s_3)}$ with $s_i \in \frac{1}{r_i}\mathbb{Z}$, modulo shift equations.

1. Write all independent structure constants in the cases $(r_1, r_2, r_3) = (\frac{1}{2}, 1, \frac{3}{2})$ and $(r_1, r_2, r_3) = (1, 2, 3)$.
2. What is the number of independent structure constants, as a function of r_1, r_2, r_3?

Exercise 7 ABDE: Degenerate structure constants

Let V_1 be a degenerate field, V_2 a diagonal field, and let V_3 be a primary field that appears in the OPE $V_1 V_2$.

1. Compute the shifts $\frac{C_{123}}{C_{123++}}$ and $\frac{C_{123}}{C_{123--}}$.
2. In which cases are these shifts zero or infinite? Interpret the results in terms of fusion rules.

3 Double Gamma function and solutions of shift equations

Exercise 8 Questions

1. Using the behaviour of the double Gamma function under $x \to x + \beta^{\pm 1}$, compute the ratio $\frac{\Gamma_\beta(x+\beta+\beta^{-1})}{\Gamma_\beta(x)}$ in two different ways, and check that the results agree.
2. Compute the residues of the double Gamma function at its poles in terms of $\Gamma_\beta(\beta)$ and the Gamma function.
3. Explicitly compute the nontrivial 3-point structure constants in the minimal models $\text{AMM}_{4,3}$ and $\text{AMM}_{5,2}$. Check the answers in [1]

Exercise 9 Convergence of the conformal block decomposition in Liouville theory

Let us show that the integral over s-channel momentums converges in the decomposition of the four-point function:

$$\left\langle \prod_{i=1}^4 V_{P_i} \right\rangle = \begin{cases} \frac{1}{2} \int_{i\mathbb{R}} dP_s \frac{C_{12s} C_{s34}}{B_s} \left| \mathcal{F}^{(s)}_{P_s} \right|^2 & \text{if } c \notin]-\infty, 1] \ , \\ \frac{1}{2} \int_{i\mathbb{R}+\epsilon} dP_s \frac{\hat{C}_{12s} \hat{C}_{s34}}{\hat{B}_s} \left| \mathcal{F}^{(s)}_{P_s} \right|^2 & \text{if } c \in]-\infty, 1] \ . \end{cases} \tag{16}$$

1. Using its integral representation:

$$\log \Upsilon_b(x) = \int_0^\infty \frac{dt}{t} \left[\left(\frac{Q}{2} - x \right)^2 e^{-2t} - \frac{\sinh^2 \left(\left(\frac{Q}{2} - x \right) t \right)}{\sinh(bt) \sinh\left(\frac{t}{b}\right)} \right] \ . \tag{17}$$

show that the Upsilon function has the asymptotic behaviour

$$\log \Upsilon_b(\tfrac{Q}{2} + P) \underset{P \to i\infty}{=} P^2 \log |P| - \frac{3}{2} P^2 + o(P^2) \ . \tag{18}$$

(Useful lemma: for any function $\phi(t)$ that is continuous at $t = 0$ and decreases fast enough at $t = \infty$, we have $\forall a > 0$, $\int_0^\infty \frac{dt}{t} \left(\phi(t) - \phi(at) \right) = \phi(0) \log a$.)

Deduce the behaviour of the combination of structure constants $\frac{C_{12s} C_{s34}}{B_s}$ as $P_s \to i\infty$.

2. Using Zamolodchikov's recursion, show that s-channel conformal blocks have the asymptotic behaviour

$$\log \mathcal{F}^{(s)}_{\Delta_s}(\Delta_i | x) \underset{\Delta_s \to \infty}{=} \Delta_s \log(16q) + O(1) \ . \tag{19}$$

3. Deduce that the integral over s-channel momentums converges for $|q| < 1$.

Behaviour of Liouville 4-point functions at coinciding points: see Exercise 3.4 in [1]

Exercise 10 Behaviour of Liouville four-point functions at coinciding points

We want to determine how the Liouville four-point function $\left\langle \prod_{i=1}^4 V_{P_i}(z_i) \right\rangle$ behaves in the limit $z_1 \to z_2$, using the s-channel decomposition (16).

1. Show that for $z_1 \to z_2$, the integral over s-channel momentums localizes near $P_s = 0$. Show that the three-point structure constants C and $\hat{C}$ behave differently near this value of the momentum.

2. Show that

$$\left\langle \prod_{i=1}^4 V_{P_i}(z_i) \right\rangle \underset{z_1 \to z_2}{\sim} \begin{cases} |z_{12}|^{\frac{Q^2}{2} - 2\Delta_{P_1} - 2\Delta_{P_2}} |\log|z_{12}||^{-\frac{3}{2}} & \text{if } c \notin]-\infty, 1] \ , \\ |z_{12}|^{\frac{Q^2}{2} - 2\Delta_{P_1} - 2\Delta_{P_2}} |\log|z_{12}||^{-\frac{1}{2}} & \text{if } c \in]-\infty, 1] \ . \end{cases} \tag{20}$$

   How does the first subleading correction behave?

3. Compare with the behaviour of a four-point function in a rational theory.

Exercise 11 Fusion rules of all unitary representations if $c > 1$

Let us investigate the fusion rules that follow from the analytically continued Liouville OPE:

$$V_{P_1} V_{P_2} \sim \frac{1}{2} \int_{i\mathbb{R}} dP \frac{C_{P_1,P_2,P}}{B_P} V_P + \sum_{P_k \in \mathcal{P}_{P_1,P_2}} \frac{\operatorname{Res}_{P=P_k} C_{P_1,P_2,P}}{B_{P_k}} V_{P_k} \ , \tag{21}$$

1. Assuming $b > 0$ so that $c \geq 25$, write the fusion rules of the Verma modules with momentums $\Pi \in]0, \frac{Q}{2}[$, i.e. of the unitary representation that do not belong to the spectrum (**??**). Check that you find

$$\mathcal{V}_\Pi \times \mathcal{V}_P = \int_{i\mathbb{R}_+} dP' \ \mathcal{V}_{P'} \ , \tag{22}$$

$$\mathcal{V}_{\Pi_1} \times \mathcal{V}_{\Pi_2} = \sum_{\Pi' \in]0, \frac{Q}{2}[\cap \left(\Pi_1 + \Pi_2 - \frac{Q}{2} - b\mathbb{N} - b^{-1}\mathbb{N}\right)} \mathcal{V}_{\Pi'} + \int_{i\mathbb{R}_+} dP' \ \mathcal{V}_{P'} \ . \tag{23}$$

2. Given a unitary Verma module $\mathcal{V}_\Delta$ of conformal dimension $\Delta > 0$, show that $\mathcal{V}_\Delta \times \mathcal{V}_\Delta$ belongs to the Liouville spectrum if $\Delta \geq \frac{c-1}{32}$, and has discrete terms if $\Delta \in]0, \frac{c-1}{32}[$.

3. If $1 < c < 25$, show that there is at most one discrete term, and determine its momentum. Deduce that the set of unitary representations is closed under fusion.

Exercise 12

In Liouville theory with DOZZ-normalized structure constants, compute the limit $\lim_{P_2 \to P_{(r,s)}} V_{P_1} V_{P_2}$ with $r, s \in \mathbb{N}^*$. Check the answer in [1]

References

[1] Sylvain Ribault. Conformal Field Theory on the plane. arXiv: 1406.4290 (3, 4)

[2] Sylvain Ribault. Exactly solvable conformal field theories. 11 2024. (1)